Int. J. Heat Mass Transfer. Vol. 10, pp. 1049–1054. Pergamon Press Ltd. 1967. Printed in Great Britain

EFFECT OF VOLUMETRIC ABSORPTION COEFFICIENT ON ROSSELAND RADIATIVE HEAT TRANSFER

R. L. SPICER

The RAND Corporation, Santa Monica, California

(*Received* 5 *March* 1965 *and in revised form* 7 *November* 1966)

Abstract—In calculating the temperature profile and heat transfer associated with an optically dense fluid it has been the practice to use Rosseland's approximation for the radiative heat flux, since its form lends itself well for use in partial differential equations. However, very little is known about the form of the absorption coefficient which is the fluid-characteristic term in the approximation. It is revealed in this analysis that in the case of uniform fluid (optically dense) flow past a flat plate, the calculated temperature profiles and heat transfer to the wall by radiation alone are sensitive to the choice of the form of the absorption coefficient.

NOMENCLATURE

- c_p, specific heat;
- f, dependent variable (T);
- h, specific enthalpy;
- k, volumetric absorption coefficient;
- n, power parameter;
- q, radiation flux;
- T, absolute temperature;
- U, fluid velocity;
- u, fluid velocity parallel to wall (U);
- v, fluid velociety perpendicular to wall (zero);
- x, distance along wall;
- y, perpendicular distance from wall.

Greek symbols

- η, similarity variable;
- ρ, fluid density;
- σ, Stefan–Boltzmann constant.

Subscripts

- n, power parameter;
- R, radiation;
- W, wall;
- ∞, free stream.

INTRODUCTION

An infinitesimal volume in an optically thick or dense gas receives radiative energy by photo absorption only from points very close to it. Therefore, for such a gas, which is also molecularly dense (number of collisions is large enough to maintain a Maxwellian distribution of excited states corresponding to the local temperature) local thermodynamic equilibrium exists; i.e. emission from this infinitesimal volume is the same as it would be from a black body in equilibrium at the same temperature. With the assumptions of an optically dense gas, local thermodynamic equilibrium, and isotropic emission, an integration over all directions of the conservation of radiation energy equation yields the Rosseland approximation for the radiation flux vector [1–3]. For the one-dimensional heat-transfer case, the Rosseland equation becomes:

$$q_R = \frac{-16\sigma T^3}{3k_R}\frac{\mathrm{d}T}{\mathrm{d}y}.$$

The purpose of this paper is to determine the effect of the volumetric absorption coefficient k_R on equilibrium radiative heat transfer q_R in an optically thick fluid, where the Rosseland approximation for q_R is used. To accomplish this, it is assumed that k_R is proportional to the absolute temperature to the nth power. This assumption is taken for mathematical convenience; however, it does represent a qualitatively correct trend, and does allow an investigation of the effect of k_R on q_R through the use of n as a parameter.

THEORETICAL CONCEPTS

The flow considered herein is an inviscid incompressible fluid passing the edge of a flat plate at a constant velocity U_∞. The flow enters at a uniform temperature T_∞ and the plate temperature is held constant at T_W (see Fig. 1).

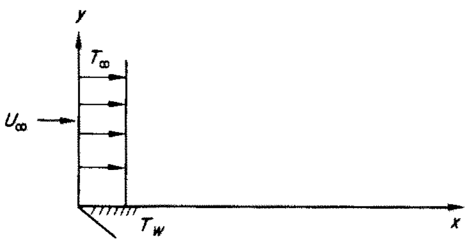

FIG. 1.

For this parallel flow, the conservation equations (energy and momentum) are uncoupled, and the conservation of mass and momentum equations, for the steady-state case, yield the trivial solution, $u = U_\infty$ and $v = 0$. Now, by assuming that the wall radiates just as the gas does, and neglecting heat transfer by conduction, the energy equation becomes [4]

$$\rho_\infty U_\infty \frac{\partial h}{\partial x} = -\frac{\partial q_R}{\partial y} \qquad (1)$$

where

$$\mathrm{d}h = c_p \, \mathrm{d}T.$$

Using the Rosseland approximation for an optically thick fluid and incorporating $k_R = aT^n = (k_{R\infty}/T_\infty^n)T^n$

$$q_R = \frac{-16\sigma T^{(3-n)}}{3a}\frac{\partial T}{\partial y}. \qquad (2)$$

By substituting equation (1) into equation (2), assuming a constant c_p, the following relation is obtained.

$$c_n \frac{\partial T}{\partial x} = (3-n)T^{(2-n)}\left(\frac{\partial T}{\partial y}\right)^2 + T^{(3-n)}\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

where

$$c_n = \frac{3\rho_\infty U_\infty c_p k_{R\infty}}{16\sigma T_\infty^n}$$

with the boundary conditions being

$$\begin{aligned} y &= 0, & T &= T_W, \\ y &= \infty, & T &= T_\infty, \qquad (4) \\ x &= 0, & T &= T_\infty. \end{aligned}$$

Equation (3) is a nonlinear second order partial differential equation which has no closed form solution. However, equations of this form have been thoroughly investigated by several authors such as Boltzmann, Crank, and others, in determining solutions to other physical problems. With the given boundary conditions, equation (3) can be transformed into a nonlinear ordinary differential equation by the use of the Boltzmann transformation or similarity variable. The form of this key transformation or similarity variable is clearly given in the works of Crank [6] and Heaslet and Alksne [5], which are concerned with the time dependent diffusion problem. In noting these works it is readily determined that the similarity variable, in this case, is given by

$$\eta_n = \frac{y}{(2\sqrt{\Gamma_n x})}$$

where

$$\Gamma_n = \frac{16\sigma T_\infty^n}{3\rho_\infty U_\infty c_p k_{R\infty}}.$$

This indeed, allows equation (3) to be transformed into the following nonlinear ordinary differential equation.

$$f'' f^{(3-n)} + (3-n) f^{(2-n)} (f')^2 + 2\eta_n f' = 0 \tag{5}$$

with the following two boundary conditions

$$f(0) = T_W \tag{6}$$

$$f(\infty) = T_\infty . \tag{7}$$

It is noted here that when $n = 3$, equation (5) reduces to the linear case which was solved by Goulard [4] wherein the normalized solution for the temperature distribution reduces to the Gauss Error Function [7] in terms of the similarity variable.

NUMERICAL SOLUTIONS

Equation (5) with the boundary conditions given by equations (6) and (7) has no closed form solution. It represents a two-point boundary value problem which may be solved by a series expansion method as in [5] or by one of several numerical integration techniques. Reference [8] displays several numerical methods of solving linear ordinary differential equations; however, these methods as such, do not guarantee that a convergence process exists when nonlinearities are present, as they are in equation (5). Therefore, by authors choice, a technique of trial and error using the calculus of finite difference was adopted. Equation (5), in finite differences form, has been programmed in Fortran for use on the IBM 7044 digital computer, such that desired solutions (f as a tabular function of η_n and n) were obtained. Since numbers had to be used in this approach, representative values of T_∞ and T_W were chosen as 10000°K and 500°K respectively. These numbers were chosen since air at extremely high temperatures [2] (associated with super-orbital re-entry and blast wave flow fields) becomes optically dense.

INTERPRETATION OF RESULTS

At this point it should be noted that the independent variable chosen (similarity variable) is dependent upon the value of n. The intent of this paper is to show the effect of n on both the temperature profiles near the wall and the heat transfer to the wall. Therefore, before any meaningful comparisons of the results can be obtained from the solutions to the differential equation (5), the dependence of η_n on n must be removed, since η_n represents a scaled distance from the wall and the scale varies with n. This dependence on n can be removed by redefining the independent variable as

$$\eta = \eta_n T_\infty^{n/2} . \tag{8}$$

Therefore the new independent variable becomes

$$\eta = \frac{y}{(2\sqrt{\Gamma x})} \tag{9}$$

where

$$\Gamma = \frac{16\sigma}{3\rho_\infty U_\infty c_p k_{R\infty}} .$$

This change in the independent variable was performed and the solutions (temperature vs. η) to equation (5) for $n = 0, 1, 2, 3, 4, 5$, and 6 are displayed in Fig. 2.

In order to get a better understanding of these results, Fig. 3 is given. This figure is a replot of Fig. 2; it shows the temperature profiles near the wall, for various n's. Here η can be considered as a scaled distance from the wall (in the y direction) at some value of x, where these profiles vary with $1/\sqrt{x}$ along the wall. Note that the larger the value of n is, the cooler the flow is near the wall since more heat is being transferred to the wall.

HEAT TRANSFER AT THE WALL

The main purpose herein is to determine the effect of the absorption coefficient through the parameter n, on heat transfer, q_R, at the wall (at $y = 0 \rightarrow \eta = 0$). In order to show the effect of the parameter n on q_R, recall equation (2):

$$q_R = \frac{-16\sigma T^{(3-n)}}{3a} \frac{\partial T}{\partial y} .$$

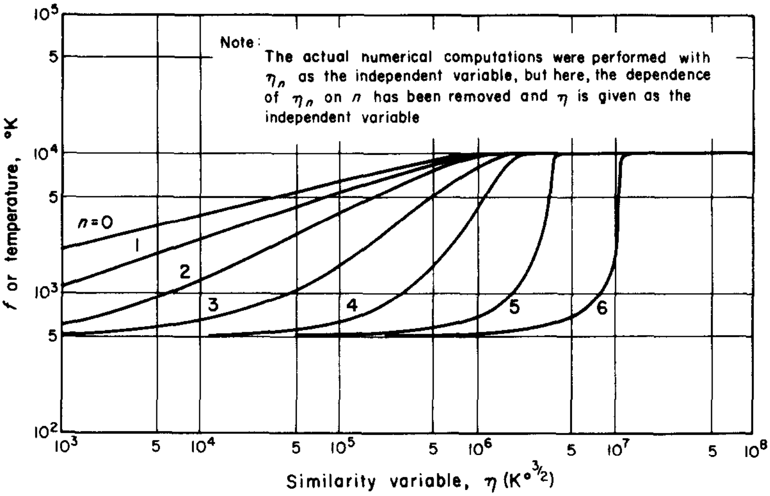

FIG. 2. Numerical solutions to the differential equation (5) for various values of the parameter n.

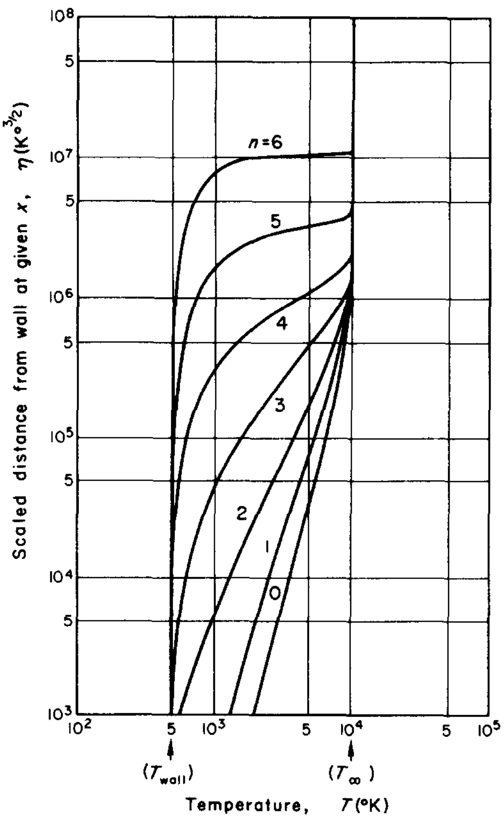

FIG. 3. Temperatire profiles for various values of n.

In terms of the similarity variable equation (2) becomes

$$\frac{q_R}{c_x} = \frac{T_\infty^n}{T^{(n-3)}}\frac{dT}{d\eta} \tag{10}$$

where

$$c_x = \frac{-8\sigma}{3k_{R\infty}(\sqrt{\Gamma x})}$$

Therefore at the wall:

$$\frac{q_{RW}}{c_x} = \frac{T_\infty^n}{T_W^{(n-3)}}\left(\frac{dT}{d\eta}\right)_{\eta=0} \tag{11}$$

Equation (11) shows the relative relationship between the heat transfer to the wall, and temperature and temperature gradient at the wall.

Figure 4 displays the temperature gradient at $\eta = 0$ vs. the power n. This figure shows a considerable variation in $(dT/d\eta)_{\eta=0}$ with n. The initial slope increases very sharply as n decreases.

Table 1 shows the total effect of n on q_{RW} [through the use of equation (11)]. Note, as n increases by a value of one, the magnitude of q_{RW}/c_x increases by as much as a factor of 3·28,

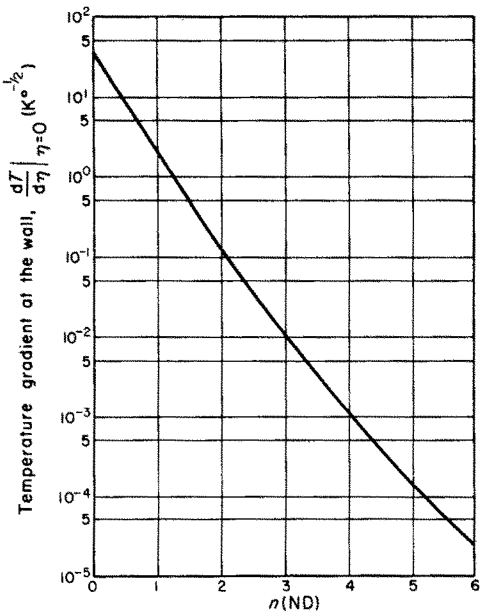

FIG. 4. Temperature gradient at wall vs. the parameter n.

Table 1. Radiative heat transfer at the wall

n	$\left(\frac{dT}{d\eta}\right)_W$	$\left(\frac{q_{RW}}{c_x}\right)$
0	$3{\cdot}500 \times 10^{1}$	$4{\cdot}375 \times 10^{9}$
1	$1{\cdot}940 \times 10^{0}$	4.850×10^{9}
2	$1{\cdot}328 \times 10^{-1}$	$6{\cdot}640 \times 10^{9}$
3	$1{\cdot}071 \times 10^{-2}$	$1{\cdot}071 \times 10^{10}$
4	$1{\cdot}100 \times 10^{-3}$	$2{\cdot}200 \times 10^{10}$
5	$1{\cdot}469 \times 10^{-4}$	$5{\cdot}875 \times 10^{10}$
6	$2{\cdot}406 \times 10^{-5}$	$1{\cdot}925 \times 10^{11}$

for the range of n's considered. From $n = 0$ to $n = 6$, the energy transferred to the wall increases by a factor of forty-four. The large values of n yield more heat transfer to the wall *by the process of radiation* than do the small values of n. This effect is to be expected since as n increases, the absorption coefficient, k_R, decreases ($T/T_\infty < 1$); therefore less energy is absorbed by the cooler portion of the flow and more energy is radiated to the wall.

CONCLUSIONS

It is seen from Figs. 3 and 4 that the temperature profile and wall temperature gradient are very much dependent upon the value of the parameter n. Table 1 indicates that the magnitude of the net energy radiated to the wall also varies with n; however, the choice of n is not as critical as the temperature gradient dependence infers. Therefore, in working a problem in radiation where the relationship $k_R \sim T^n$ is used, the degree of the precision with which n is chosen should depend on which flow quantity is of interest.

REFERENCES

1. R. GOULARD, Fundamental equations of radiation gas dynamics, Purdue University A&ES 62-4, p. 50 (March 1962).
2. R. GOULARD, Similarity parameters in radiation gas dynamics, Purdue University A&ES 62-8, p. 32 (September 1962).
3. R. VISKANTA, Heat transfer in thermal radiation absorbing and scattering media, ANL-6170, p. 87 (May 1960).
4. R. GOULARD, The transition from black body to Rosseland formulations in optically thick flows, *Int. J. Heat Mass Transfer* **6** (10), 873 (1963).
5. M. A. HEASLET and A. ALKSNE, Diffusion from a fixed surface with a concentration-dependent coefficient, *J. Soc. Ind. Appl. Math.* **9** (4), 588 (1961).
6. J. CRANK, *The Mathematics of Diffusion*, pp. 148 and 162. Clarendon Press, Oxford (1956).
7. M. JAKOB, *Heat Transfer*, Vol. 1, p. 252. Wiley and Sons, New York (1957).
8. L. FOX, *Numerical Solution of Two Point Boundary Problems*. Clarendon Press, Oxford (1957).

Résumé—Dans le calcul du profil de température et du transport de chaleur associé à un fluide optiquement dense, on a couramment utilisé l'approximation de Rosseland pour le flux de chaleur par rayonnement, puisque sa forme la rend toute indiquée pour l'emploi dans des équations aux dérivées partielles. Cependant, on connaît très peu la forme du coefficient d'absorption qui est le terme caractéristique du fluide dans l'approximation. La théorie montre que dans le cas d'un écoulement uniforme du fluide (optiquement dense) le long d'une plaque plane, les profils de température calculés et le transport de chaleur à la paroi le rayonnement seul sont sensibles au choix de la forme du coefficient d'absorption.

Zusammenfassung—Bei der Berechnung des Temperaturprofils und des Wärmeüberganges in einem optisch dichten Medium ist es üblich, die Rosseland-Näherung für die Strahlung zu verwenden, da sich ihre Form in die partielle Differentialgleichung fügt. Jedoch ist sehr wenig über die Form des Absorptionskoeffizienten bekannt, der in der Näherung den für das Medium charakteristischen Ausdruck darstellt. Es zeigt sich in dieser Analyse, dass im Fall einheitlicher Strömung eines (optisch dichten) Mediums entlang einer ebenen Platte die errechneten Temperaturprofile und der Wärmeübergang an die Wände durch Strahlung allein, auf die Wahl der Form des Absorptionskoeffizienten empfindlich reagieren.

Аннотация—В обычной практике расчета распределения температуры и теплообмена в оптически плотной среде применяется приближение Росселанда для лучистого потока тепла, т.к. его форма удобна для решения дифференциальных уравнений в частных производных. Однако данных о коэффициенте абсорбции, входящем в аппроксимацию, в виде члена, характеризующего жидкость, недостаточно. В результате анализа выяснилось, что в случае обтекания плоской пластины однородной жидкостью (оптически плотной) расчетные профили температуры и теплообмена излучением на стенке зависят от выбранной формы коэффициента абсорбции.